

# Math 247A Lecture 5 Notes

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January 15, 2020

## 1 Banach Space Properties of Lorentz Spaces

### 1.1 Proof of completeness, duality, and more

**Theorem 1.1.** For  $1 < p < \infty$  and  $1 \leq q \leq \infty$ ,

$$\|f\|_{L^{p,q}}^* \sim_{p,q} \sup \left\{ \left| \int f(x)g(x) dx \right| : \|g\|_{L^{p',q'}}^* \leq 1 \right\}.$$

Thus,  $\|\cdot\|_{L^{p,q}}^*$  is equivalent to a norm, with respect to which  $L^{p,q}(\mathbb{R}^d)$  is a Banach space. Moreover, for  $q \neq \infty$ , the dual of  $L^{p,q}$  is  $L^{p',q'}$ , under the natural pairing.

*Proof.* Last time, we saw that it suffices to prove the equivalence for functions of the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n}$$

with  $F_n$  measurable, pairwise disjoint, and  $\|f\|_{p,q}^* \sim \|2^n |F_n|^{1/p}\|_{\ell_n^q} \sim 1$ . Last time, we showed that  $RHS \lesssim LHS$  by testing it on  $g = \sum 2^m \mathbb{1}_{E_m}$  with  $E_m$  measurable, pairwise disjoint, and  $\|g\|_{L^{p',q'}}^* \sim \|2^m |E_m|^{1/p'}\|_{\ell_m^{q'}} \lesssim 1$ . Let's show that  $LHS \lesssim RHS$ .

Compare: in the case of  $L^q(\mathbb{R}^d)$ , we take  $g = \frac{|f|^{q-1} \operatorname{sgn} f}{\|f\|_q^{q-1}}$ . Here, we take

$$g = \sum \left( 2^n |F_n|^{1/p} \right)^{q-1} |F_n|^{-1/p'} \mathbb{1}_{F_n}$$

Check:

$$\begin{aligned} \int f(x)g(x) dx &= \sum_n 2^n \left( 2^n |F_n|^{1/p} \right)^{q-1} |F_n|^{-1/p'} |F_n| \\ &= \sum_n 2^{nq} |F_n|^{q/p} \\ &= \|2^n |F_n|^{1/p}\|_{\ell_n^q}^q \end{aligned}$$

$$\begin{aligned} &\sim (\|f\|_{L^{p,q}}^*)^q \\ &\sim 1. \end{aligned}$$

It remains to show that  $\|g\|_{L^{p',q'}}^* \lesssim 1$ . By the proposition which evaluates the norm as a dyadic sum,

$$\left(\|g\|_{L^{p',q'}}^*\right)^{q'} \sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} |\{x : N \leq g(x) < 2N\}|^{q'/p'}$$

Note that  $\{x : g \sim N\} = \bigcup_{n \in S_N} F_n$ , where  $S_N = \{n \in \mathbb{Z} : 2^{n(q-1)} |F_n|^{q/p-1} \sim N\} = \{n \in \mathbb{Z} : |F_n| \sim [N 2^{-n(q-1)}]^{p/(q-p)}\}$ .

$$\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left( \sum_{n \in S_N} |F_n| \right)^{q'/p'}$$

This is a dyadic sum, so we can pull the exponent inside (by our lemma).

$$\begin{aligned} &\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \sum_{n \in S_N} |F_n|^{q'/p'} \\ &\sim \sum_{n \in \mathbb{Z}} \left( 2^{n(q-1)} |F_n|^{q/p-1} \right)^{q'} |F_n|^{q'/p'} \end{aligned}$$

Use  $q' = q/(q-1)$ .

$$\begin{aligned} &\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q'(q/p-1/p)} \\ &\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q/p} \\ &\sim (\|f\|_{L^{p,q}}^*)^q \\ &\sim 1. \end{aligned}$$

The RHS defines a norm  $\|\cdot\|$ . To see that  $L^{p,q}$  equipped with this norm is a Banach space, one uses the usual Riesz-Fischer argument.

Step 1: If  $\|f_n\| \in L^{p,q}$  are such that  $\sum \|\cdot\| < \infty$ , then there exists a function  $f \in L^{p,q}$  such that  $f = \sum f_n$  in  $\|\cdot\|$ .

Step 2: For a Cauchy sequence  $\{f_n\}_{n \geq 1} \subseteq L^{p,q}$ , we pass to a subsequence so that  $\|f_{k_{n+1}} - f_{k_n}\| < \frac{1}{2^n}$ . So by Step 1,

$$f_{k_n} = f_{k_1} + \sum_{j=2}^n f_{k_j} - f_{k_{j-1}} \xrightarrow{\|\cdot\|} f.$$

For  $1 \leq q < \infty$ , we want to show that the dual of  $L^{p,q}$  is  $L^{p',q'}$ . Let  $\ell : L^{p,q} \rightarrow \mathbb{R}$  be a linear functional, so  $\|\ell(f)\| \lesssim \|f\|_{L^{p,q}}^*$ . For  $f = \mathbb{1}_E$  with  $E$  of finite measure,

$$\ell(\mathbb{1}_E) \lesssim |E|^{1/p}.$$

So the measure  $E \mapsto \ell(\mathbb{1}_E)$  is absolutely continuous with respect to Lebesgue measure. So there exists a  $g \in L_{\text{loc}}^1$  such that

$$\ell(\mathbb{1}_E) = \int g(x) \mathbb{1}_E(x) dx.$$

As  $\ell$  is linear, we get

$$\ell(f) = \int f(x)g(x) dx$$

for all simple functions  $f \in L^{p,q}$ .

- Claim 1: Boundedness of  $\ell$  on simple functions yields  $g \in L^{p',q'}$ .
- Claim 2: Simple functions are dense in  $L^{p,q}$  if  $q \neq \infty$ .

Given these two claims, we get  $\ell(f) = \int f(x)g(x) dx$  for all  $f \in L^{p,q}$ . Thus, the dual of  $L^{p,q}$  is  $L^{p',q'}$ .

Proof of Claim 1: It is enough to show that if  $g = \sum 2^m \mathbb{1}_{E_m}$  with  $E_m$  measurable, pairwise disjoint, we have

$$\|2^m |E_m|^{1/p'}\|_{\ell^{q'}} \lesssim 1.$$

Choose  $f = \sum_{|m| \leq M} \left(2^m |E_m|^{1/p'}\right)^{q'-1} |E_m|^{-1/p} \mathbb{1}_{E_m}$ . Then

$$\ell(f) = \int f(x)g(x) dx \sim \|2^m |E_m|^{1/p'}\|_{\ell_{|m| \leq M}^{q'}}^{q'} \quad , \quad \|f\|_{L^{p,q}} \sim \|2^m |E_m|^{1/p'}\|_{\ell_{|m| \leq M}^{q'}}^{q'/q}.$$

We have  $\ell(f) \lesssim \|f\|_{L^{p,q}}^*$ , so

$$\|2^m |E_m|^{1/p'}\|_{\ell_{|m| \leq M}^{q'}} \lesssim \|2^m |E_m|^{1/p'}\|_{\ell_{|m| \leq M}^{q'}}^{1/(q-1)}.$$

This gives

$$\|2^m |E_m|^{1/p'}\|_{\ell_{|m| \leq M}^{q'}} \lesssim 1$$

uniformly in  $M$ . So  $g \in L^{p',q'}$ .

Proof of Claim 2: Consider  $f \geq 0$  and look at  $f_m = f \mathbb{1}_{\{2^m \leq f < 2^{m+1}\}}$ . For  $2^{m+n} \leq k \leq 2^{m+1+n} - 1$ , let

$$E_{m,n}^k = f^{-1} \left( \left( \frac{k}{2^n}, \frac{k+1}{2^n} \right) \right), \quad \varphi_{m,n} = \sum_{k=2}^{2^{m+n+1}-1} \frac{k}{2^n} \mathbb{1}_{E_{m,n}^k}.$$

Then  $0 \leq f_m - \varphi_{m,n} \leq \frac{1}{2^n}$ . First choose  $\varepsilon > 0$ . If we look at  $\|f\|_{L^{p,q}}^* \| \|f_m\|_{L^p} \|_{\ell^q}$ , only finitely many terms matter, so we can truncate the series. This lets us estimate  $\|f_m - \varphi_{m,n}\|_{p',q'}^*$ , as any large numbers we get will be multiplied by our small step size,  $\frac{1}{2^n}$ .  $\square$